THE MATHEMATICS OF CHAOS

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Abstract

Seemingly random, chaotic dynamic systems have state variables that move about in a non-periodic, bounded fashion. The sensitivity to initial conditions of chaotic signals also holds an interesting pattern. A seemingly tiny change in the values of initial conditions, can greatly affect the values of the output. This means that cross-correlation of two chaotic signals from the same source can be very low. By nature, chaotic systems are required to be non-linear and dynamic systems.The behaviour of dynamic systems was modeled using MATLAB,with pertinent non-linearities.Various mathematical aspects were studied.Since differential equations are an integral part of modeling dynamic systems, finding solutions to differential equations using MATLAB programming was also done.

Keywords: Chaos, attractors, bifurcation diagram and non-linear dynamic systems.

I. INTRODUCTION

Chaos theory describes the behavior of certain nonlinear dynamical systems that may exhibit dynamics that are highly sensitive to initial conditions. As a result of this sensitivity, which manifests itself as an exponential growth of perturbations in the initial conditions, the behavior of chaotic systems appears to be random. This happens even though these systems are deterministic, meaning that their future dynamics are fully defined by their initial conditions, with no random elements involved. This behavior is known as deterministic chaos, or simply chaos.

Chaotic behavior has been observed in the laboratory in a variety of systems including electrical circuits, lasers, oscillating chemical reactions, fluid dynamics, and mechanical and magneto-mechanical devices. Observations of chaotic behavior in nature include the dynamics of satellites in the solar system, the time evolution of the magnetic field of celestial bodies, population growth in ecology, the dynamics of the action potentials in neurons, and molecular vibrations. Everyday examples of chaotic systems include weather and climate.

Systems that exhibit mathematical chaos are deterministic and thus orderly in some sense; this technical use of the word chaos is at odds with common parlance, which suggests complete disorder. A related field of physics called quantum chaos theory studies systems that follow the laws of quantum mechanics. Recently, another field, called relativistic chaos, has emerged to describe systems that follow the laws of general relativity.

II. CHAOTIC DYNAMICS

For a dynamical system to be classified as chaotic, it must have the following properties:

- It must be sensitive to initial conditions,
- It must be topologically mixing, and
- Its periodic orbits must be dense.

Sensitivity to initial conditions means that each point in such a system is arbitrarily closely approximated by other points with significantly different future trajectories. Thus, an arbitrarily small perturbation of the current trajectory may lead to significantly different future behavior.

Sensitivity to initial conditions is popularly known as the "butterfly effect", so called because of the title of a paper given by Edward Lorenz in 1972 in the American Association for the Advancement of Science in Washington, D.C. entitled “Predictability: Does the Flap of a Butterfly's Wings in Brazil set off a Tornado in Texas?”. The flapping wing represents a small change in the initial condition of the system, which causes a chain of events leading to large-scale phenomena. Had the butterfly not flapped its wings, the trajectory of the system might have been vastly different.

Topologically mixing means that, the system will evolve over time so that any given region or open set of its phase space will eventually overlap with any other given region. Here, "mixing" is really meant to correspond to the standard intuition: the mixing of colored dyes or fluids is an example of a chaotic system.

III. ATTRACTORS

Some dynamical systems are chaotic everywhere but in many cases chaotic behaviour is found only in a subset of phase space. The cases of most interest arise when the chaotic behaviour takes place on an attractor, since then a large set of initial conditions will lead to orbits that converge to this chaotic region. While most of the motion types mentioned above give rise to very simple attractors, such as points and circle-like curves called limit cycles, chaotic motion gives rise to what are known as strange attractors, attractors that can have great detail and complexity. The Lorenz attractor is a 3-dimensional structure corresponding to the long-term behavior of a chaotic flow.

The map shows how the state of a dynamical system (the three variables of a three-dimensional system) evolves over time in a complex, non-repeating pattern.

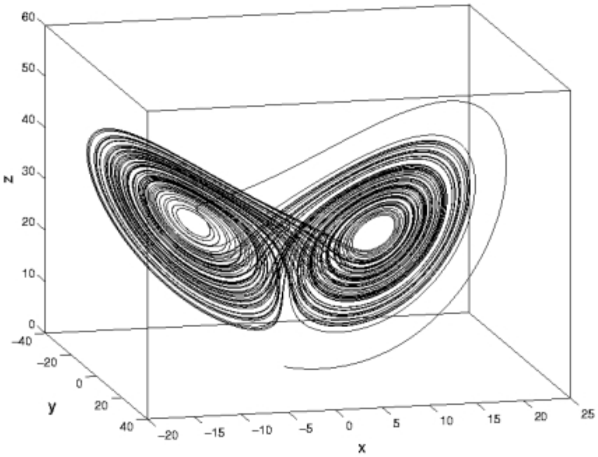

Fig. 1. Lorenz Attractor

IV. PHASE PORTRAIT

A phase portrait is a geometric representation of the trajectories of a dynamical system in the phase plane. Each set of initial conditions is represented by a different curve, or point. Phase portraits are an invaluable tool in studying dynamical systems. They consist of a plot of typical trajectories in the state space. This reveals information such as whether an attractor, a repellor or limit cycle is present for the chosen parameter value. The concept of topological equivalence is important in classifying the behavior of systems by specifying when two different phase portraits represent the same qualitative dynamic behavior.

Phase portraits are obtained by plotting one of the output states of the system against another with both states evolving in terms. In phase portrait a single loop indicates a single period. Multiple routes intersecting with each other indicate that many periods are present.

V. BIFURCATION DIAGRAM

Bifurcation theory is the mathematical study of how and when the solution to a problem changes from there only being one possible solution, to there being more than one, which is called a bifurcation.Most commonly used in the mathematical study of dynamical systems, a bifurcation occurs when a small smooth change made to the parameter values (the bifurcation parameters) of a system causes a sudden 'qualitative' or topological change in its long-term dynamical behavior.

An example is the bifurcation diagram of the logistic map:

$$x_{n+1} = rx_n(1 - x_n).$$

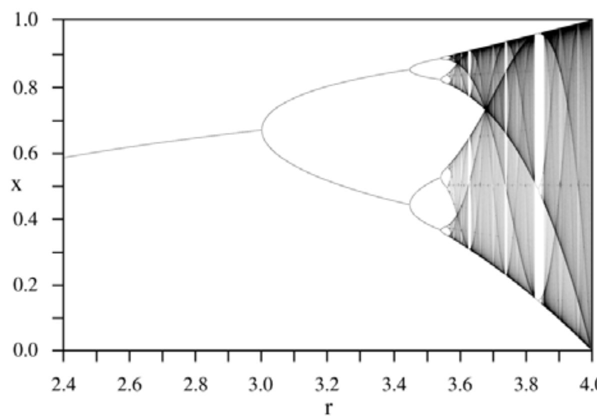

Fig. 2. Bifurcation Plot

The bifurcation parameter r is shown on the horizontal axis of the plot and the vertical axis shows the possible long-term population values of the logistic function. Only the stable solutions are shown here, there are many other unstable solutions which are not shown in this diagram. The bifurcation diagram nicely shows the forking of the possible periods of stable orbits.

A feature seen in these diagrams is that chaotic regions are interspersed with periodic windows. Bifurcation diagrams also exhibit scaling i.e. if we zoom one region it resembles the whole. These bifurcation diagrams help to identify regions of chaos and periodic behavior that help in proper design of chaotic systems.

VI. APPLICATIONS

Chaos theory is applied in many scientific disciplines: mathematics, biology, computer science, economics, engineering, finance, philosophy, physics, politics, population dynamics, psychology, and robotics. Chaos theory is also currently being applied to medical studies of epilepsy, specifically to the prediction of seemingly random seizures by observing initial conditions.

Recently Chaos theory has been applied to Radio over Fiber Technology in Optical Communication.

VII. CHAOTIC SIGNALS AND SYSTEMS

A. Introduction

Many non-linear dynamical systems do not follow simple, regular, and predictable trajectories. These systems evolve in a random-like, but well-defined, fashion. As long as the process involved is nonlinear, even a simple deterministic model may develop such complex behavior-chaos. Many disciplines have recognized the benefits of chaotic systems and try to exploit their particular properties. Today, the challenge from "how to avoid" the chaotic behavior has been changed to "how to control" or "how to exploit" it.

VIII. CHAOTIC SIGNAL

It can be difficult to tell from data whether a physical or other observed process is random or chaotic, because in practice no time series consists of pure 'signal.' There will always be some form of corrupting noise, even if it is present as round-off or truncation error. Thus any real time series, even if mostly deterministic, will contain some randomness.

All methods for distinguishing deterministic and stochastic processes rely on the fact that a deterministic system always evolves in the same way from a given starting point. Thus, given a time series to test for determinism, one can:

1. pick a test state;
2. search the time series for a similar or 'nearby' state; and
3. Compare their respective time evolutions.

Define the error as the difference between the time evolution of the 'test' state and the time evolution of the nearby state. A deterministic system will have an error that either remains small (stable, regular solution) or increases exponentially with time (chaos). A stochastic system will have a randomly distributed error.

Essentially all measures of determinism taken from time series rely upon finding the closest states to a given 'test' state (i.e., correlation dimension, Lyapunov exponents, etc.). To define the state of a system one typically relies on phase space embedding methods. Typically one chooses an embedding dimension, and investigates the propagation of the error between two nearby states. If the error looks random, one increases the dimension. If you can increase the dimension to obtain a deterministic looking error, then you are done. Though it may sound simple it is not really. One complication is that as the dimension increases the search for a nearby state requires a lot more computation time and a lot of data (the amount of data required increases exponentially with embedding dimension) to find a suitably close candidate. If the embedding dimension (number of measures per state) is chosen too small (less than the 'true' value) deterministic data can appear to be random but in theory there is no problem choosing the dimension too large – the method will work. Practically, anything approaching about 10 dimensions is considered so large that a stochastic description is probably more suitable and convenient anyway.

IX. HAOTIC SYSTEMS

A. Autonomous Dynamical Systems

Dynamical systems can be classified as autonomous and non-autonomous dynamical systems. The governing equations of a non-autonomous dynamical system depend on time, while the governing system equation of an autonomous dynamical system is independent of the time. The non-autonomous dynamical systems will not be considered here, because:

- Any non-autonomous system can be transformed to an equivalent higher order autonomous system, and
- The chaotic systems used in chaotic communications schemes are autonomous dynamical systems.

B. Chaotic Steady - State

At least a third order continuous time nonlinear system or a first-order discrete time nonlinear system with non-invertible map is required to get a chaotic behavior. Chaos is bounded steady-state behavior that is not an equilibrium point, not periodic, and not quasi-periodic. In the time domain the chaotic signals are random-like but well defined signals which can be predicted only in the short-term. In the frequency domain, due to the non-periodicity characterizing the evolution in the time domain, chaotic signals has broad-band "noise-like" spectrum.

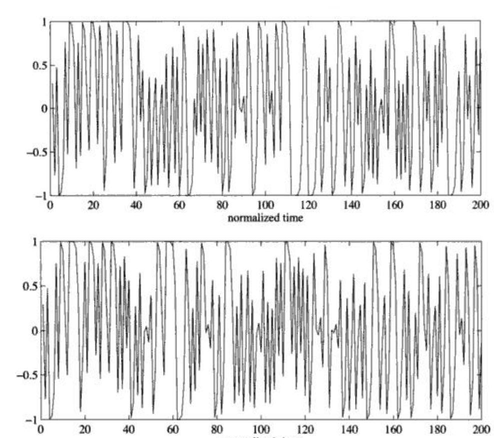

Fig. 1.3. Waveform of chaotic signal from the first order discrete time nonlinear system with same initial conditions

X. NON LINEAR SYSTEM

Necessary conditions for Chaos:

- Non-linearity
- Dynamic behavior
- Governing state equations or differential equations

Approaches that can be used for non-linearity:

- Power series model
- Saleh model
- Differential Equation model
- Soft/hard limiter model

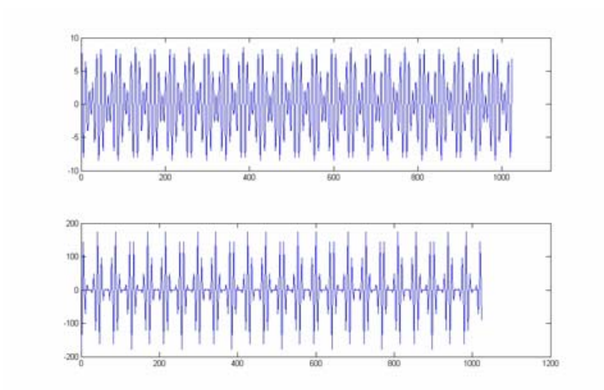

Fig. 4. Power Series Model

Consider a system defined by the following

$d^3y/dt^3 + 7d^2y/dt^2 + 3|y(t)|dy/dt + 9y(t) = 4\,e^{(-t/2)}$ for t¨20

$d^3y/dt^3 + 7d^2y/dt^2 + 3\,y(t)dy/dt + 9y(t) = 4\,e^{(-t/2)}$ for t>=20

The resulting attractor for the above system is plotted in Figure 5.

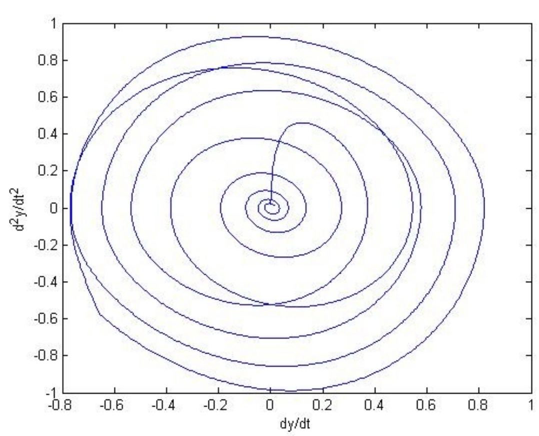

Fig. 5. System Attractor

XI. CONCLUSION

To get chaotic behavior, at least a third-order nonlinear dynamical system has to be considered in the continuous-time domain, or a first order discrete-time system with non-invertible map has to be used.

Chaotic signals are aperiodic and their spectrum is continuous and usually broadband. Since their statistical properties may be designed and controlled, chaos find application in (broadband) communication.

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